Binomial coefficients are (almost) never powers

Venkatramani Rajgopal
Hochschule Mittweida, University of Applied Sciences

11 January, 2016

1 Introduction

This is a epilogue to Bertrand's postulate on Binomial Coefficients.

Bertrands postulate.
For every $n \geq 1$ there is some prime number p with $n < p \leq 2n$.

In 1892 J.Sylvester strengthened Bertrands postulate in the following way;

If $n \geq 2k$, then at least one of the numbers $n, n-1, ...n-k+1$ has a prime divisor p greater than k.

Note that for $n = 2k$ we obtain precisely Bertrands postulate. In 1934, Erdos gave a elementary Book Proof of Sylvesters result, running along the lines of his proof of Bertrands postulate.

He mentioned an equivalent way of stating Sylvesters theorem:
The binomial coefficient,

$$\binom{n}{k} = \frac{n(n-1)...(n-k+1)}{k!} \qquad (n \geq 2k)$$

always has a prime factor $p > k$.

With this observation we analyse when is $\binom{n}{k}$ equal to power m^l.
We see that there are infinitely many solutions for $k = l = 2$, i.e., there are infinitely many solutions of $\binom{n}{2} = m^2$.
We observe that if $\binom{n}{2}$ is a square, then so is $\binom{(2n-1)^2}{2}$. To see this, let $n(n-1) = 2m^2$. So by substitution we get

$$(2n-1)^2((2n-1)^2-1) = (2n-1)^2 4n(n-1) = 2(2m(2n-1))^2.$$

So we have:

$$\binom{(2n-1)^2}{2} = (2m(2n-1))^2$$

Beginning with $\binom{9}{2} = 6^2$ we thus obtain many solutions. The next one is $\binom{289}{2} = 204^2$. [*by substituting n=9 we have* $(2n-1)^2 = (2*9-1)^2 = 289$]
For $k = 3$ it is known that $\binom{n}{3} = m^2$ has a unique solution with $n = 50, m = 140$. But for $k \geq 4$ and $l \geq 2$ we do not have any further solutions. Erdos proved this by the following argument:

2 Theorem

The Equation $\binom{n}{k}$=m^l has no integer solutions with $l \geq 2$ and $4 \leq k \leq (n-4)$.

Proof. We may assume $n \geq 2k$ since $\binom{n}{k} = \binom{n}{n-k}$. If the theorem is false then it follows that $\binom{n}{k} = m^l$ This proof by contradiction proceeds in the below four steps.

2.1 Step 1

By Sylvester's theorem, $\binom{n}{k}$ has a prime factor $p > k$ of . We have that $\binom{n}{k} = m^l$ which can be written as

$$\frac{n(n-1).....(n-k+1)}{k(k-1).....1} = m^l$$

is divisible by p. Since $p > k$,then p can't be a divisor of the denominator $k(k-1).....1$. Which implies that the numerator $n(n-1).....(n-k+1)$ is indeed divisible by p. So we have

$$\frac{n(n-1).....(n-k+1)}{k(k-1).....1} = m^l \,\vdots\, p \Rightarrow \frac{n(n-1).....(n-k+1)}{k(k-1).....1} = m^l \,\vdots\, p^l$$

Since p is not a divisor of $k(k-1).....1$ then we can write:

$$n(n-1).....(n-k+1) \,\vdots\, p^l$$

Only one of $n-i \,\vdots\, p^l$, since $l \geq 2$ we make the following observation

$$\boldsymbol{n \geq p^l > k^l \geq k^2} \tag{1}$$

2.2 Step 2

We rewrite the $(n-j)$ factors of the numerator in the form:

$$\boldsymbol{(n-j) = a_j m_j^l} \tag{2}$$

Where $0 \leq j \leq k-1$ and a_j is not divisible by any l-th power. By **(Step 1)** we know that a_j has only prime divisors less than or equal to k. We want to show $a_i \neq a_j$ when $i \neq j$. We assume the opposite, that there exist i, j such that $a_i = a_j$ and $i \neq j$, we can assume $i < j$ (otherwise $j > i$) .Then we have

$$\begin{aligned} i < j &\implies n - i > n - j \\ &\implies a_i m_i^l > a_j m_j^l \\ &\implies m_i^l > m_j^l \implies m_i > m_j \implies m_i \geq m_j + 1 \end{aligned}$$

On the other hand:

$$(0 \leq i, j \leq k-1 \text{ and } i < j) \implies k > j - i = (n-i) - (n-j) = a_j(m_i^l - m_j^l) \geq a_j((m_j+1)^l - m_j^l) \tag{3}$$

Now :

$$(m_j+1)^l - m_j^l = \sum_{k=0}^{l} \binom{l}{k} m_j^{l-k} - m_j^l = [\binom{l}{0} m_j^l + \binom{l}{1}) m_j^{l-1} + \cdots + 1] - m_j^l > \binom{l}{1} m^{l-1} = l m_j^{l-1}$$

Plugging the above inequality in (3) we conclude

$$k > a_j((m_j+1)^l - m_j^l) > l a_j m_j^{l-1} \tag{4}$$

We know that $l \geq 2$, so:

$$(l/2) \geq 1 \implies (l-1) \geq (l/2) \implies m_j^{l-1} \geq m_j^{l/2}$$

Since $j \leq k-1$ we can also write:

$$a_j l m_j^{l-1} \geq l(a_j m_j)^{l/2} = (n-j)^{1/2} \geq (n-(k-1))^{1/2}$$

which leads to

$$l(a_j m_j)^{l/2} \geq l((n-(k-1))^{1/2}) \tag{5}$$

.

From our assumption, $n \geq 2k \implies k \leq (n/2) \implies n-k+1 \geq n-n/2+1 = n/2+1$. Furthermore:

$$(n-(k-1))^{1/2}) \geq (n/2+1)^{1/2} \implies l((n-(k-1))^{1/2})) \geq l((n/2+1)^{1/2}) \tag{6}$$

And since $l \geq 2$ we have

$$l((n/2+1)^{1/2}) > l(n/2)^{1/2} = (l^2/2)^{1/2} n^{1/2} = n^{1/2}$$

Therefore we can say

$$l((n/2+1)^{1/2}) > n^{l/2} \tag{7}$$

Now combining equations 4, 5 , 6 and 7 we get:

$$\begin{aligned} k &> l a_j m_j^{l-1} \\ &\geq l(a_j m_j)^{1/2} \geq (n-(k-1))^{1/2} \\ &\geq n^{1/2} \end{aligned}$$

which is a contradiction to $n > k^2$, so our assumption that there exist i, j such that $a_i = a_j$ and $i \neq j$ is wrong and therefore $a_i \neq a_j$ whenever $i \neq j$ i.e, a_j's are all distinct.

2.3 Step 3

In this step we prove a_i's are the integers 1,2,....k in some order. Since we know that they all are distinct, it suffices to prove that,

$$a_0 a_1 \ldots a_{k-1} \text{ divides } k!$$

Substituting $n-j = a_j m_j^l$, from Equation 2, into the equation $\binom{n}{k} = m^l$, we obtain,

$$\begin{aligned} n(n-1)\ldots(n-k+1) &= a_0 m_0^l a_1 m_1^l \ldots\ldots a_{k-1} m_{k-1}^{l-1} \\ &= (a_0 a_1 \ldots a_{k-1})(m_0 m_1 \ldots m_{k-1})^l \\ &= k! m^l \end{aligned}$$

Now cancelling common factors of $m_0 m_1 \ldots m_{k-1}$ and m yields,

$$a_0 a_1 \ldots a_{k-1} u^l = k! v^l \tag{8}$$

where $\gcd(u, v) = 1$. We want to show that $v = 1$. If $v \neq 1$ then it has a prime factor $p \leq k$. Equation (8) tells us that since $\gcd(u, v) = 1$ and u^l cannot be divisible by p then $a_0 a_1 \ldots a_{k-1}$ has to be divisible by p, so p has to be less than or equal to k and therefore p appears somewhere in the product $k! = k(k-1)\ldots 1$.

By Legendre's Theorem we know that the exponent of p in $k!$ is

$$\sum_{i \geq 1} \left\lfloor \frac{k}{p^i} \right\rfloor$$

Since $n(n-1)\dots(n-(k-1)) = a_0a_1\dots a_{k-1}(m_0m_1\dots m_{k-1})^l = k!m^l$ then p also appears in the product $n(n-1)\dots(n-(k-1))$. Next we estimate the exponent of p in this product. Let $i > 0$ and let's assume that there are s multiples $b_1 < b_2 < \dots < b_s$ of p^i among $n, (n-1), \dots, (n-(k-1))$ where $0 \le i \le k-1$ and $0 \le s \le k$, i.e $b_s = s \cdot p^i$, $b_1 = 1 \cdot p^i$. Furthermore we have

$$\begin{aligned} b_s &= b_1 + b_s - b_1 \\ &= b_1 + p^i \cdot s - p^i \\ &= b_1 + (s-1)p^i \end{aligned}$$

Since $b_1 < b_2 < \dots < b_s$ are multiples of p^i among $n, (n-1), \dots, (n-(k-1))$ we have

$$(s-1)p^i = b_s - b_1 \le n - (n-k+1) = k-1 \implies s = \frac{k-1}{p^i} + 1$$

which implies

$$s \le \left\lfloor \frac{k-1}{p^i} \right\rfloor + 1 \le \left\lfloor \frac{k}{p^i} \right\rfloor + 1 \tag{9}$$

So for each i the number of multiples of p^i among $n, ...n-k+1$ and hence among the $a_j's$ is bounded by $\left\lfloor \frac{k}{p^i} \right\rfloor + 1$.

This implies that the exponent of p in $a_0a_1....a_{k-1}$ is at most

$$\sum_{i=1}^{l-1}(\left\lfloor \frac{k}{p^i} \right\rfloor + 1) \tag{10}$$

The argument is the same as in Legendre's thoerem the difference here is that the sum stops at $i = l-1$, since the $a_j's$ contain no l-th powers. Extracting v^l from equation (8) we have

$$v^l = \frac{a_0a_1....a_{k-1}u^l}{k!}$$

Knowing that the exponent of a fraction is the difference of exponents ($\frac{a^m}{a^n} = a^{m-n}$) we have the following estimation for the exponent of v^l

$$exp(v^l) = \sum_{i=1}^{l-1}(\left\lfloor \frac{k}{p^i} \right\rfloor + 1) - \sum_{i\ge1}\left\lfloor \frac{k}{p^i} \right\rfloor = \sum_{i=1}^{l-1}\left\lfloor \frac{k}{p^i} \right\rfloor - \sum_{i\ge1}\left\lfloor \frac{k}{p^i} \right\rfloor + \sum_{i=1}^{l-1} 1 \le l-1 \tag{11}$$

which is a contradiction to the fact that v^l has exponent l. So our assumption that $v \ne 1$ is wrong. So $v = 1$ and therefore $u = 1$. So we can write $k! = a_0a_1\dots a_{k-1}$. Indeed, since $k \ge 4$ one of the $a_i's$ must be equal to 4, i.e $a_i = 4 = 2^2 = 2^l$, which is a contradiction to the fact that that $a_i's$ contain no squares. This suffices to settle the case $l = 2$. So we now assume that $l \ge 3$

2.4 Step 4

Since $k \ge 4$ and $k! = a_0a_1.....a_k - 1$ then for some i_1, i_2, i_3 we have $a_{i_1} = 1, a_{i_2} = 2, a_{i_3} = 4$, that is

$$\begin{aligned} n - i_1 &= a_{i1}m_1^l = m_1^l \\ n - i_2 &= a_{i2}m_2^l = 2m_2^l \\ n - i_3 &= a_{i3}m_3^l = 4m_3^l \end{aligned}$$

We claim that $(n-i_2)^2 \neq (n-i_1)(n-i_3)$. Assume the opposite that, $(n-i_2)^2 = (n-i_1)(n-i_3)$ and let

$$n - i_2 = b$$
$$n - i_1 = b - x$$
$$n - i_3 = b + y$$

where $0 < |x|, |y| < k$. Hence we have

$$b^2 = (b-x)(b+y) \implies (y-x)b = xy$$

where $x = y$ is not possible because in the contrary we would have

$$b^2 = (b-x)(b+y) = (b-x)(b+x) = b^2 - x^2 \implies x^2 = 0$$

which is not possible because $|x| > 0$. By part **(1)**

$|xy| = b|y-x| \geq b > n-k \geq k^2 \geq (k-1)^2 \geq |xy|$, which is incorrect. Therefore our assumption $(n-i_2)^2 = (n-i_1)(n-i_3)$ is incorrect. That means $(2 \cdot m_2^l)^2 \neq m_1^l \cdot 4 \cdot m_3^l$. Dividing by 4 we have, $(m_2^l) \neq m_1^l m_3^l \implies m_2^2 \neq m_1 m_3$. Without losing generality we assume $m_2^2 > m_1 m_3$ (otherwise $m_2^2 < m_1 m_3$) so we have $\implies m_2^2 \geq m_1 m_3 + 1$.

Using the fact that $n^2 - (n-k+1)^2 = 2(k-1)n - (k-1)^2$ we write

$$\begin{aligned}
2(k-1)n &> 2(k-1)n - (k-1)^2 \\
&= n^2 - (n-k+1)^2 \\
&> (n-i_2)^2 - (n-i_1)(n-i_3) \\
&= (2m_2^l)^2 - 4(m_1 m_3)^l \\
&= 4[m_2^{2l} - (m_1 m_3)^l] \\
&\geq 4[(m_1 m_3 + 1)^l - (m_1 m_3)^l] \\
&\geq 4l m_1^{l-1} m_3^{l-1}
\end{aligned}$$

Multiplying both sides by $m_1 m_3$ we have,

$$2(k-1)n m_1 m_3 > 4l m_1^l m_3^l = l(n-i_1)(n-i_3) > l(n-k+1)^2 \tag{12}$$

Plugging $l \geq 3$ at equation (1) we get

$$n > k^l \geq k^3 > 6k \implies k < \frac{n}{6} \tag{13}$$

Having the above observation we keep estimating the right side of inequation (12)

$$l(n-k+1)^2 > 3(n - \frac{n}{6})^2 > 2n^2 \tag{14}$$

Combination of (12) and (14) yieds

$$2(k-1)n \cdot m_1 \cdot m_3 > l(n-k+1)^2 > 2n^2$$

by dividing with $2n$ both sides we have

$$(k-1)m_1 m_3 > n \tag{15}$$

Observe next that

$$n - i = a_i m_i^l \implies n > a_i m_i^l$$

taking l-th root of both sides we have

$$n^{1/2} > a_i^{1/l} m_i$$

So

$$m_i \leq n^{1/l} \leq n^{1/3} \implies m_1 m_3 \leq n^{1/3} \cdot n^{1/3} = n^{2/3}$$

And we obtain

$$m_1 m_3 \leq n^{2/3} \tag{16}$$

Multiplying by k both sides of (16) and using (15) we obtain

$$kn^{2/3} \geq km_1 m_3 > (k-1)m_1 m_3 > n,$$

by taking third power and dividing with n we have $n < k^3$ which is contradiction to equation to (12).

Which contradicts $n \geq k^3$. Therefore our assumption that $\binom{n}{k} = m^l$ for $l \geq 3$ is wrong, so there is no solution to $\binom{n}{k} = m^l$ for $l \geq 3$ and $k \geq 4$.

□

References

[1] Martin Aigner, Gnter M. Ziegler. *Proofs from the book. Fourth Edition.* Springer 2013